# STABILITY REGION OF CLOSED-LOOP PILOT-VEHICLE SYSTEM FOR FLY-BY-WIRE AIRCRAFT WITH LIMITED ACTUATOR RATE

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***Abstract:***

*The category-II PIO (Pilot Induced Oscillations) caused by actuator rate limitation of fly-by-wire airplanes will badly threaten the flight safety. The stability regions of closed-loop pilot-vehicle (CLPV) system with rate limited actuator were studied in this paper to assess stability of such CLPV system. The augmented state variables were introduced to segregate the rate limited element from the primary system in order to build the saturation nonlinear model of CLPV system. To get the maximal stability region, firstly, the estimation of the stability region of CLPV system is transformed into convex optimization problem; secondly, the Schur complement lemma is applied to transform the convex optimization problem into linear matrix inequalities(LMIs) formulations; finally, the ellipsoidal stability region estimation algorithm is obtained. The time-domain simulation results show that the estimated stability region is slightly conservative and within the real stability region of CLPV system. The category II PIO of static unstable fly-by-wire airplanes is a kind of much rapid divergent oscillation instead of limit cycle oscillation. Moreover, the stability region is distinctly influenced by the pilot controlling gains and the actuator rate limitation value, the proposed stability region method exhibits clear physical concept and intuitionistic results.*

## 1. Introduction

With the development of complex multidisciplinary airplane design process, proper early estimates of aerodynamic characteristics are essential [1]. To enhance the performance of modern fighters, they are usually designed with relaxed or negative static stability. Broadened static stability boundary technology is adopted in flight design and aircrafts are schemed out to be static unstable or nearly

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critical stable. The fly-by-wire (FBW) control system is used to compensate the static stability of the aircrafts; thereby the aircraft can be normally controlled by the pilot. Essentially, the augmentation of stability is obtained through additional deflection of the control surfaces. But the authority of the FBW control system is limited by physical limits on displacements and rates of control surface motions. So when the physical limit has been reached, the stability of the aircraft cannot be compensated by FBW control system, thus PIOs (pilot induced oscillations) will mostly occur when the aircraft is maneuvered by the pilot. And this kind of PIO are usually defined as category II PIO [2]. The United State Department of Defense (DoD) defines PIO [3] as 'sustained or uncontrollable oscillations resulting from the efforts of the pilot to control the aircraft. Essentially, PIO is a destabilization phenomenon of the Closed-Loop Pilot-Vehicle System. Due to its uncertain and hard-to-forecasting oscillatory or divergent property, flight safety may be severely threatened by PIO. The FBW control system is used to increase the probability of PIO. The deflection rate of control surfaces should be as small as possible but it needs to satisfy other requirements of flight dynamics. In order to predict PIOs in the aircraft design procedure, lots of methods are applied, such as describing function method [4], OLOP criterion [5], Gap criterion [6], robust stability method [7], time-domain Neal-Smith criterion [8], unified theory method [9] and so on. Since PIO is closely related to the stability of the Closed-Loop Pilot-Vehicle (CLPV) System, PIO can be predicted by investigation on the stability region of CLPV system. Stability region of one system is defined as a region in the state space (meaning) that, if the states of this system after being disturbed keep remaining in this region, the system will be stable; contrarily, if the system states are out of this region after being disturbed, then the system state will go far away from the region and the system will be unstable. With regard to CLPV system, providing the rate limiter of the actuator has not been triggered, the CLPV system is stable and the flight states are in the stability region of the system, while once the rate limiter has been activated, the stability region of the CLPV system will be changed, accordingly, if the flight states are in the stability region of the CLPV system, it will be stable. Otherwise, it will be unstable and PIO will occur.

By means of the investigation of stability region of the CLPV system, on the one hand, the safety of current flight condition can be evaluated to judge the danger of destabilization; on the other hand, the design of flight control law can be guided so as to make the stability region as large as possible. Consequently, the occurrence probability of PIO is as low as possible. Reference [10] provides one method to estimate the stability region of the CLPV system, the results show that: ① The PIOs of statically-stable aircraft is one kind of limit cycles. When the rate limiter is activated and the saturation is slight, the CLPV system remains stable. ② The stability region calculated through this method is very conservative, mainly due to its non-optimal Lyapunov energy function. So this paper deals with the strength of ① and ② , the stability region calculation method of CLPV system with rate limited actuator towards statically unstable FBW aircraft. In order to get as large stability region as possible to avoid the conservation of stability region, the calculation of the stability region was transformed into the solution process of linear matrix inequalities. The influence of pilot control gain and rate saturation value on the stability region is researched in detail.

## 2. Modeling of the CLPV system with actuator rate limit

### 2.1 Framework of the CLPV system

A simplified longitudinal pitch attitude tracking CLPV system is considered in Fig. 1. It is composed of pilot model, rate limited actuator model and aircraft dynamical model.

The synchronous control model [11] is used to simulate the pilot model, which is given by

$$G_{\mathrm{p}}(s) = K_{\mathrm{p}} \tag{1}$$

where $K_{\mathrm{p}}$ denotes the gain of pilot model.

The rate limited actuator model [12] is shown in Fig. 1.

The state space model of the bare aircraft dynamics is given by

$$\begin{aligned} \dot{x}_p &= A_p x_p + B_p u_p \\ y_p &= C_p x_p + D_p u_p \end{aligned} \tag{2}$$

where $x_p = [v, \alpha, q, \theta]^T$ is state vector; $v$ denotes flight speed; $\alpha$ denotes angle of attack; $q$ denotes pitch rate; $\theta$ denotes pitch angle; $u_p = [\delta]$ is input vector; $\delta$ denotes elevator angle; $y_p = [v, \alpha, q, \theta]^T$ is output vector.

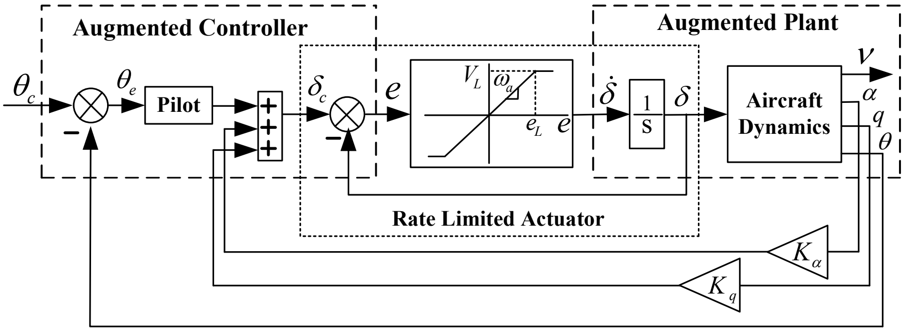



*Figure 1. Closed-loop pilot-vehicle system of pitch attitude tracking*

**2.2 State equation model of the CLPV system**

To calculate the stability region, the state space model of the CLPV system must be built firstly. The process of modeling is as follows:

Step 1: establish dynamic model of the augmented plant dynamics

As shown in Fig. 1, the augmented plant consists of aircraft dynamics and integral element of rate limited actuator. The state variables of the bare aircraft dynamics $v, \alpha, q, \theta$ are combined with the output variable of rate limited actuator $\delta$, the state vector of augmented plant becomes $x_m = [v, \alpha, q, \theta, \delta]^T$, state variables $\alpha, q, \theta$ are combined with $\delta$, so the output vector of the augmented plant becomes $y_m = [\alpha, q, \theta, \delta]^T$. Correspondingly, $u_m = [\dot{\delta}]$ is the input vector of the augmented plant. So the single input, four outputs state model of the augmented plant is given by

$$\begin{aligned} \dot{x}_m &= A_m x_m + B_m u_m \\ y_m &= [\alpha \quad q \quad \theta \quad \delta]^T = C_m x_m \end{aligned} \tag{3}$$

where

$A_m = \begin{bmatrix} A_p & B_p \\ 0 & 0 \end{bmatrix}$, $B_m = [0_{1\times4}, I_{1\times1}]^T$, $C_m = [0, I_{4\times4}]$.

Step 2: establish dynamic model of the augmented controller

The augmented controller dynamics are made up of pilot dynamics and the gain of rate limited actuator, denoted as $u_r$, such that $u_m = [\dot{\delta}] = sat(u_r)$ can be obtained from Fig. 1. The four inputs single output state model of the augmented controller is given by

$$\begin{aligned} u_r &= -\omega_a \cdot [-k_\alpha \quad -k_q \quad K_p \quad 1] \cdot y_m \\ &= -\omega_a \cdot [-k_\alpha \quad -k_q \quad K_p \quad 1] \cdot C_m x_m \end{aligned} \tag{4}$$

Step 3: build state equation model of the CLPV system

According to Step 1 and Step 2, the state equation model of the CLPV can be obtained with $\dot{\delta}$ as the input variable, $u_r$ as the output variable and $\tilde{x} = [v, \alpha, q, \theta, \delta]^T$ as the state vector.

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{A}\tilde{x} + \tilde{B}\dot{\delta} \\ \tilde{y} &= \tilde{C}\tilde{x} \\ \dot{\delta} &= sat(\tilde{y}) \end{aligned} \tag{5}$$

where $\tilde{A} = A_m$, $\tilde{B} = B_m$,
$\tilde{C} = -\omega_a \cdot [-k_\alpha \quad -k_q \quad K_p \quad 1] \cdot C_m$, $u_r = \tilde{y}$

## 3. Stability region estimation based on LMIs

Consider a single input system as below

$$\dot{x} = Ax + Bu \tag{6}$$

where $x \in R^n$ is the state variable; $u \in R$ is the control input; $A$ and $B$ are real matrices. Furthermore, pair $(A, B)$ is assumed to be controllable.

The control input is assumed to be in the compact set $\Omega \subset R$ according to the physical limit of the actuator:

$$\Omega = \{u \in R, -u_0 \le u \le u_0\} \tag{7}$$

where $u_0$ is the saturation value of input. The saturation function is defined as follows:

$$sat(Cx) = sign(Cx)\min\{u_0, |Cx|\} \tag{8}$$

where $C \in R^{1\times n}$ is the feedback gain matrix, sign() is signum function. If saturation control law is $u = -sat(Cx)$, the model of closed-loop system is given by

$$\dot{x} = Ax - Bsat(Cx) \tag{9}$$

When the control is unsaturated, $x \in S(C, u_0)$ can be described as follows:

$$S(C, u_0) = \{x \in R^n, -u_0 \le Cx \le u_0\} \tag{10}$$

System (9) is linearized as:

$$\dot{x} = (A - BC)x \tag{11}$$

The ellipsoidal stability region [13] of system (11) is given by

$$\varepsilon(P,1) = \{x \in R^n : x^T P x < 1\} \tag{12}$$

where $P \in R^{n\times n}$ denotes symmetrical positive definite matrix.

**Theorem 1** [13] For system (6), $u_0 = 1$, if the control input is bounded, the maximum stability region of system (6) satisfying saturation function (8) is $\varepsilon(P,1)$, which can be obtained by solving the following convex optimization problem in the variable $W = W^T \in R^{n\times n}$:

$$\begin{cases} \max \log\det W \\ s.t. \quad W = W^T \\ CWC^T \le 1 \\ (A-BC)W + W(A-BC)^T < 0 \end{cases} \tag{13}$$

where $W = P^{-1}$.

**Corollary 1** For system (6), $u_0 > 0$, if the control input is bounded, the stability region of system (6) satisfying saturation function (8) denoted as $\varepsilon(P,1)$ can be computed by solving the following convex optimization [14] problem in the variable $W = W^T \in R^{n\times n}$:

$$\begin{cases} \max \log\det W \\ s.t. \quad W = W^T \\ CWC^T \le u_0^2 \\ (A-BC)W + W(A-BC)^T < 0 \end{cases} \tag{14}$$

where $W = P^{-1}$.

Using Schur complements, $\max \log\det W$ is equal to the extremum existence problem of $\gamma$ in the inequality [15] given by

$$\gamma V \ge W^{-1} \Leftrightarrow \begin{bmatrix} \gamma V & I \\ I & W \end{bmatrix} \ge 0, \gamma > 0, f_{\inf}(\gamma) \tag{15}$$

where $V$ denotes symmetric positive definite matrix; $f_{\inf}(\cdot)$ denotes infimum function.

To simplify the solving process, the constraint $CWC^T \le u_0^2$ can be transformed into normal LMI through Schur complements, given by

$$CWC^T \le u_0^2 \Leftrightarrow \begin{bmatrix} u_0^2 & C \\ C^T & W^{-1} \end{bmatrix} \ge 0 \tag{16}$$

Finally, according to (14), (15), (16), when $u_0 > 0$, the maximal ellipsoid stability region can be computed by solving the following LMIs

$$\begin{cases} f_{\inf}(\gamma) \\ s.t.\quad W = W^T \\ \begin{bmatrix} \gamma V & I \\ I & W \end{bmatrix} \geq 0 \\ V > 0 \\ \gamma > 0 \\ \begin{bmatrix} u_0^2 & C \\ C^T & W^{-1} \end{bmatrix} \geq 0 \\ (A-BC)W + W(A-BC)^T < 0 \end{cases} \tag{17}$$

So once the state equation model of the CLPV system described in Fig. 1 has been set up, the stability region can be calculated by solving LMIs similar to inequalities (17).

## 4. Numerical example

For the CLPV system shown in Fig. 1, some parameters are given by

$$A_{\mathrm{p}} = \begin{bmatrix} -0.033102 & 0.38576 & -0.20764 & -0.56119 \\ -0.015511 & -1.2588 & 1.0114 & -0.00247 \\ 0.008121 & 0.95415 & 0.65786 & -0.000441 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$B_{\mathrm{p}} = \begin{bmatrix} -0.5193 \\ -0.05243 \\ -11.085 \\ 0 \end{bmatrix}, \qquad C_{\mathrm{p}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$D_p = [0]$, $\omega_a = 20 rad/\mathrm{s}$, $V_{\mathrm{L}} = 30^{\circ}/\mathrm{s}$; $K_{\mathrm{p}} = -1.1$.

### 4.1 Conservatism analysis of the stability region

Solving LMIs (17) according to the above parameters, five-dimensional hyper-ellipsoid stability region of the CLPV system with rate limited actuator can be obtained. In order to get a direct view, the five-dimensional hyper-ellipsoid stability region is projected onto ($\alpha, q$) subspace. In order to assess the conservatism of the stability region obtained by solving the LMIs, three initial state points are selected from the interior, exterior and boundary of the stability region to conduct time-domain simulation. The two-dimensional stability region and three state trajectories corresponding to three initial state points are shown in Fig. 2. State trajectory 1 and 3 are within the stability region all the time. State trajectory 2 converges to the equilibrium point within the stability region in the end, which is to say, the stability region being calculated is conservative and within the real stability region of the CLPV system.

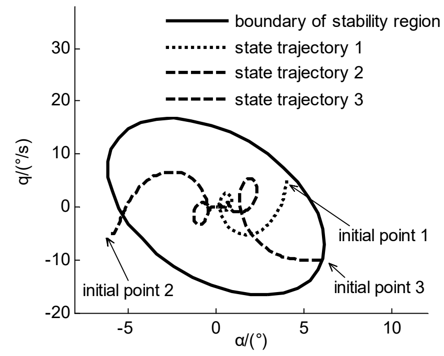
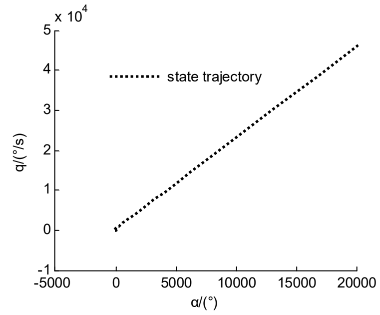


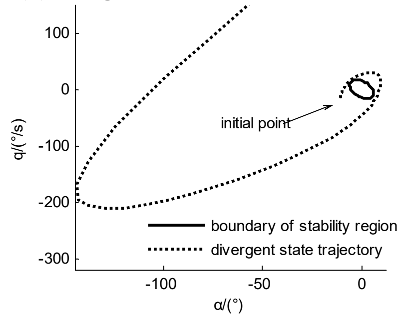

*Figure 2. Projection of stability region onto ($\alpha, q$) subspace and three state trajectories*

### 4.2 Analysis of instability feature of the CLPV system

The research in [10] shows that category II PIOs of statically-stable aircraft are limit cycles, and the output waveform of the rate limited actuator possesses triangular feature. But for statically-unstable aircraft, the category II PIOs lead to divergent oscillations, which can be concluded from Fig. 3.




*(a) Original simulation result*




*(b) Partial enlarged drawing of (a)*

*Figure 3. Divergent state trajectory of initial point outside the stability region*

Figure 4 shows the output of the actuator when PIOs occur at the level corresponding to Figure 3, but in the 5th second, the deflection angle of actuator exceeds 60 degrees, in fact, the maximal deflection angle of actuator in real aircraft is not more than 30 degrees. Figure 5 shows the time history of attack angle $\alpha$ when PIOs occur, at 3.5th second, the attack angle has exceeded stalling angle far and away, at the same time in Fig. 4 the deflection of actuator is 20 degrees, smaller than the maximal deflection of the actuator. The time domain simulation is reasonable. In these figures, the main reason why the angle of attack and pitch rate are beyond their limits is mainly to show the whole trajectory and its divergence.

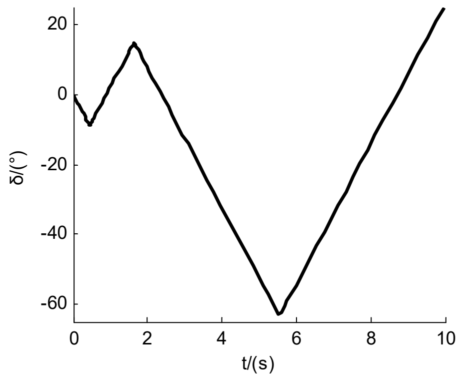



*Figure 4. Output of actuator*

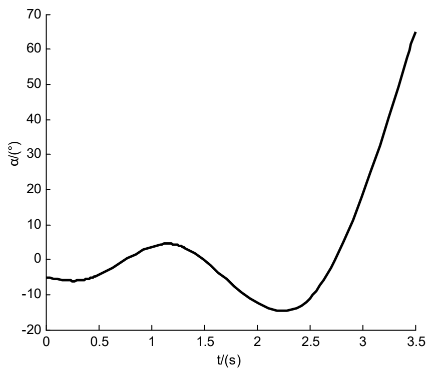
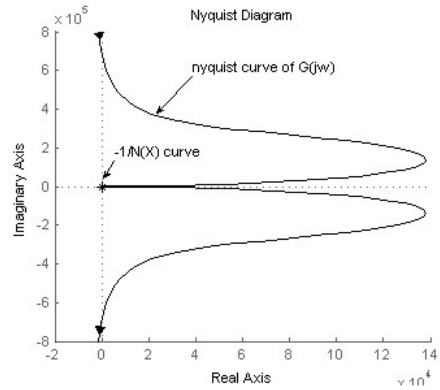


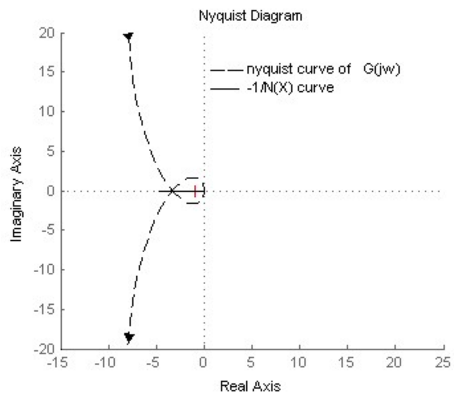

*Figure 5. Time history of the angle of attack*

### 4.3 Stability analysis of the CLPV system in frequency domain

For the system shown in Fig. 1, the describing function (DF) method can be applied to judge its stability. The DF of saturation element is given by $N(X)$. $X$ denotes the input signal entering the saturation element. The model of the remnant system is presented in Fig. 1 except for saturation element which is given by transfer function $G(j\omega)$. Similar to the Nyquist criterion in linear systems, the stability criterion of saturation nonlinear systems based on DF is that: if the frequency characteristic curve of $G(j\omega)$ does not surround the negative reciprocal DF curve of $-1/N(X)$, the system is stable; otherwise, the system is unstable. The negative reciprocal DF curve of saturation element and Nyquist curve of $G(j\omega)$ on the same complex plane is plotted in Fig. 6. The asterisk "*" denotes the negative reciprocal DF curve of saturation element in Fig. 6 (a), while in Fig. 6 (b), it is denoted by the solid line.



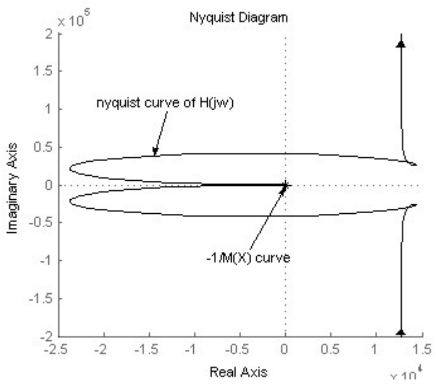

*(a) Entire simulation result*



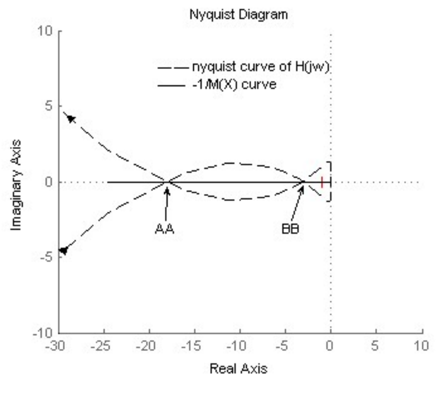

*(b) Partial enlarged drawing of (a)*

*Figure 6. $G(j\omega)$ and $-1/N(X)$ curves of pilot-vehicle system*

Figure 6 shows that the negative reciprocal DF curve of $-1/N(X)$ is contained in the Nyquist curve of $G(j\omega)$, that is to say, once the rate limitation of the actuator has been triggered, the oscillation amplitude of the system will always increase until the system becomes divergent. It is obvious that the frequency domain analysis is consistent with the time domain simulation shown in Fig. 3 and Fig. 5. As an example of statically stable aircraft in reference [10], Fig. 7 is given to be in contrast with Fig. 6. Similar to Fig. 6, asterisk "*" and the solid line denote the negative reciprocal DF curve of saturation element in Fig. 7 (a) and Fig. 7 (b), respectively. In Fig. 7, the point "AA" denotes the self-oscillation point of the system, that is to say, when the rate limitation of the actuator is activated, the limit cycles of the system may occur, which is consistent to the time domain simulation in reference [9].




*(a) Entire simulation result*




*(b) Partial enlarged drawing of (a)*

*Figure 7. $H(j\omega)$ and $-1/M(X)$ curves of pilot-vehicle system*

### 4.4 Influence factors on the stability region

The activation of rate limited element of the actuator is the main factor to trigger category II PIO, while rough control of the pilot on the aircraft is the direct reason to activate the rate limited element of the actuator, so the pilot gain and the value of rate limitation are two important influence factors on the stability region.

Calculating the stability region with the value of rate limitation $V_{L1} = 40^\circ/s$, the $(\alpha, q)$ subspace projection of stability region can be obtained. Comparison of the magnitude of the stability region with the one whose value of rate limitation is $V_L = 30^\circ/s$ is given in Fig. 8. Apparently, the larger the value of rate limitation is, the larger the magnitude of the stability region becomes.

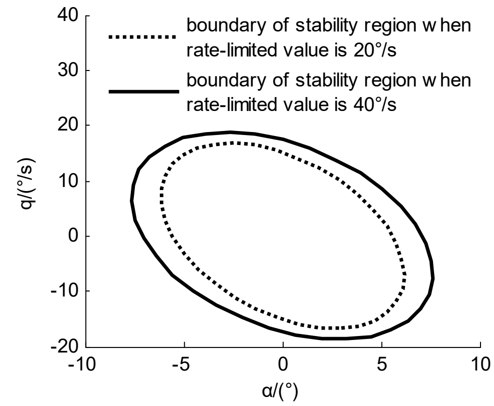



*Figure 8. Stability regions of different rate-limited values*

Calculating the stability region with the pilot gain $K_{p1} = -2$, the $(\alpha, q)$ subspace projection of stability region is given in Fig. 9.

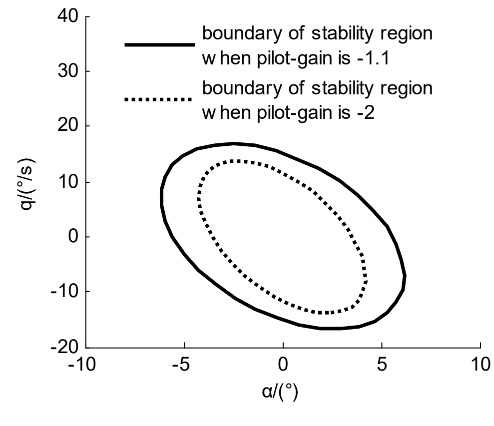



*Figure 9. Stability regions of different pilot-gain values*

At the same time, Fig. 9 also shows the comparison of the magnitude of the stability region when the pilot gains are $K_p = -1.1$ and $K_{p1} = -2$, respectively.

Apparently, the larger the pilot gain is, the smaller the magnitude of the stability region is. So, if the pilot gain is large enough, the PIO may occur with a small attack angle disturbance, which is similar to the case in linear system where the stability margin may be decreased due to increasing system gain.

## 5. Conclusion

A stability region calculating method of the CLPV system with a rate limiting actuator is proposed in this paper. The conservatism of the stability region estimated by the proposed method and the stability of the CLPV system relevant to category II PIO are analyzed. The detailed conclusions are as follows:

1) Some frequency domain analysis methods can judge the stability but cannot give the stable ranges of the states of the CLPV system directly. On the contrary, calculating stability region can provide the stable ranges of the states of the CLPV system. So the stability of the CLPV system can be evaluated more efficiently through synthesizing the two methods.

2) The analysis of the specific example indicates that the stability region calculated by solving LMIs is conservative, so how to find the true stability region of the CLPV system is one key factor in predicting the Category II of PIO analysis method in project application.

3) PIOs of statically stable aircrafts are limiting cycle oscillations, while for statically unstable aircrafts, PIOs are divergent oscillations. The flight safety may be disrupted seriously because once the rate limiter of actuator has been triggered, the motions of aircraft will diverge quickly, and then the pilots cannot effectively control the aircraft to avoid flying accident within a short time interval.

4) The pilot control gain and rate limiting value of the actuator are distinct influencing factors on the stability region. On the one hand, using high capacity actuators can increase the rate limiting value of the actuator but it is a disadvantage for high performance fighters; on the other hand, the pilots are to manipulate the airplanes softly but not roughly. When PIO occurs, they should give up the manipulation to avoid disastrous accidents.

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